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Convergence theorems for asymptotically pseudocontractive mappings in the intermediate sense

H. Zegeye *, M. Robdera, B. Choudhary

Department of Mathematics, University of Botswana, Pvt. Bag 00704, Gaborone, Botswana

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ABSTRACT

In this paper, we prove a strong convergence of Ishikawa scheme to a uniformly L-Lipschitzian and asymptotically pseudocontractive mappings in the intermediate sense. No compactness assumption is imposed either on T or C, and computation of intersection of closed convex sets C_n and Q_n for each $n \geq 1$ is not required. We also obtain convergence results in this direction for asymptotically strict pseudocontractive mappings in the intermediate sense. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear mappings.

1. Introduction and preliminaries

Let H be a real Hilbert space; $\emptyset \neq C \subset H$; a mapping $T : C \to C$ is said to be *nonexpansive*, if for all $x, y \in C$ we have $\|Tx - Ty\| \leq \|x - y\|$. It is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ and $\lim k_n = 1$ such that $\|T^nx - T^ny\| \leq k_n\|x - y\|$ for all integers $n \geq 0$ and all $x, y \in C$. Clearly, every nonexpansive mapping is asymptotically nonexpansive with sequence $k_n = 1$, $\forall n \geq 0$. There are, however, asymptotically nonexpansive mappings, which are not nonexpansive (see e.g., [1]).

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] in 1972 and has been studied by several authors (see e.g., [3–8]). Goebel and Kirk proved that if C is a nonempty closed convex and bounded subset of a uniformly convex Banach space (more general than Hilbert space), then every asymptotically nonexpansive self-mapping of C has a fixed point.

T is said to be *asymptotically nonexpansive in the intermediate sense* if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|T^nx - T^ny\| - \|x - y\|\right) \leq 0. \tag{1.1}$$

Observe that if we define

$$\tau_n := \max\left\{0, \sup_{x,y\in C}\left(\|T^nx - T^ny\| - \|x - y\|\right)\right\}, \tag{1.2}$$

then $\tau_n \to 0$ as $n \to \infty$. It follows that (1.1) reduces to

$$\|T^nx - T^ny\| \leq \|x - y\| + \tau_n, \quad \forall x, y \in C. \tag{1.3}$$

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck et al. [9]. It is known [10] that if C is a nonempty closed convex subset of a Hilbert space H, and T is asymptotically nonexpansive in the intermediate sense, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings.

* Corresponding author.
E-mail addresses: habtuzh@yahoo.com (H. Zegeye), robdera@yahoo.com (M. Robdera), choudha2000@yahoo.com (B. Choudhary).

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Let C be a subset of real Hilbert space H and $T : C \to C$ be any map. T is said to be *asymptotically pseudocontractive* if there exists $\{k_n\} \subset [1, \infty)$ such that the inequality

$$\langle T^n x - T^n y, x - y\rangle \leq k_n \|x - y\|^2, \tag{1.4}$$

holds for all $x, y \in C$ and for all integers $n \geq 1$. It is trivial to see from inequality (1.4) that every asymptotically nonexpansive mapping is asymptotically pseudocontractive. T is called *uniformly L-Lipschitzian* if there exists $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|, \forall x, y \in C$ and for each integer $n \geq 1$ and T is called *completely continuous* if $T(K)$ is relatively compact for all bounded sets $K \subset C$.

The class of asymptotically pseudocontractive mappings was introduced by Schu [7] (see also [11]). In [5], Rhoades gave an example to show that the class of asymptotically pseudocontractive mappings contains properly the class of asymptotically nonexpansive mappings; see [5] for more details. In 1991, Schu [7] established the following classical result.

Theorem SC (*[7]*). *Let H be a real Hilbert space, $C \subset H$ be nonempty closed bounded and convex. Let T be a completely continuous uniformly L-Lipschitzian and asymptotically pseudocontractive self-map of C with sequences $\{k_n\} \subset [1, \infty)$. Let $\{x_n\}$ be a sequence defined by $x_1 \in C$ and*

$$\begin{cases} y_n = \beta_n T^n x_n + (1 - \beta_n)x_n, \\ x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Assume that the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$; *where $q_n := 2k_n - 1$ for each $n \geq 1$.*
(ii) $a \leq \alpha_n \leq \beta_n \leq b$ *for some $a > 0$ and some $b \in (0, L^{-2}[\sqrt{1 + L^2} - 1])$.*

Then the sequence $\{x_n\}$ generated by (1.5) converges strongly to a common fixed point of $\{T_i, i = 1, 2, \ldots, N\}$.

But we observe that the assumption T is *completely continuous*, that is, $T(C)$ is relatively compact, is severe restriction. This brings us to the following question:

Question 1. *Is it possible to obtain strong convergence of scheme (1.5) (called Ishikawa type scheme) to a fixed point of asymptotically pseudocontractive mappings without the assumption that T be completely continuous?*

Next, we recall the definition of asymptotically pseudocontractive mapping in the intermediate sense.

A mapping $T : C \to C$ is said to be *asymptotically pseudocontractive mapping in the intermediate sense* if

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\langle T^n x - T^n y, x - y\rangle - k_n\|x - y\|^2\right) \leq 0, \tag{1.6}$$

where $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$. Observe that if we put

$$\nu_n := \max\left\{0, \sup_{x,y\in C}\left(\langle T^n x - T^n y, x - y\rangle - k_n\|x - y\|^2\right)\right\}, \tag{1.7}$$

then we get that $\nu_n \to 0$ as $n \to \infty$ and (1.6) is reduced to the following:

$$\langle T^n x - T^n y, x - y\rangle \leq k_n\|x - y\|^2 + \nu_n, \quad \forall n \geq 1,\ x, y \in C. \tag{1.8}$$

Moreover, we obtain that (1.8) is equivalent to:

$$\|T^n x - T^n y\|^2 \leq (2k_n - 1)\|x - y\|^2 + \|(I - T^n)x - (I - T^n)y\|^2 + 2\nu_n, \quad \forall n \geq 1, x, y \in C. \tag{1.9}$$

If $\nu_n = 0$ for each $n \geq 1$, then the class of asymptotically pseudocontractive mappings in the intermediate sense is reduced to the class of asymptotically pseudocontractive mappings.

The class of asymptotically pseudocontractive mappings in the intermediate sense was introduced by Qin et al. [12]. In [12], Qin et al. established the following classical result.

Theorem QCK (*[12]*). *Let H be a real Hilbert space, $C \subset H$ be nonempty closed bounded and convex. Let T be a uniformly L-Lipschitzian and asymptotically pseudocontractive self-map of C in the intermediate sense with sequences $\{k_n\} \subset [1, \infty)$ and $\{\nu_n\} \subset [0, \infty)$ defined as in (1.9). Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \beta_n T^n x_n + (1 - \beta_n)x_n, \\ x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geq 1, \end{cases} \tag{1.10}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Assume that the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$ *where $q_n := 2k_n - 1$ for each $n \geq 1$;*
(ii) $a \leq \alpha_n \leq \beta_n \leq b$ *for some $a > 0$ and some $b \in (0, L^{-2}[\sqrt{1 + L^2} - 1])$.*

Then the sequence $\{x_n\}$ generated by (1.10) converges weakly to a fixed point of T.

But note that the convergence obtained is *weak convergence*. This brings us to our second question:

Question 2. *Is it possible to obtain strong convergence of Ishikawa type scheme (1.10) to a fixed point of asymptotically pseudocontractive mappings in the intermediate sense?*

In this connection, Qin et al. [12] established the hybrid Ishikawa algorithm for uniformly L-Lipschitzian asymptotically pseudocontractive mappings in the intermediate sense with sequences $\{k_n\} \subset [1, \infty)$ and $\{\nu_n\} \subset [0, \infty)$ as follows:

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n T^n x_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n T^n z_n; \\ C_n = \left\{ u \in C : \|y_n - u\|^2 \le \|x_n - u\|^2 + \alpha_n\theta_n + \alpha_n\beta_n\big(q_n\beta_n + \beta_n^2 L^2 + \beta_n - 1\big)\|T^n x_n - x_n\|^2 \right\}; \\ Q_n = \{u \in C : \langle x_1 - x_n, x_n - u\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \ge 1, \end{cases} \tag{1.11}$$

where $q_n := 2k_n - 1$ and $\theta_n := q_n([1+\beta_n(q_n-1)]-1)M + 2(q_n+1)\nu_n$ for each $n \ge 1$ and for some $M > 0$ and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. They proved that the sequence $\{x_n\}$ defined by (1.11) converges strongly to $P_{F(T)}x_0$, where P_C is the metric projection from H into C, provided that $F(T) \neq \emptyset$. You may also see strong convergence results of Zegeye and Shahzad [13,14] for ϕ-asymptotically nonexpansive mappings and for a finite family of asymptotically nonexpansive mappings and semigroups in Banach spaces. But we observe that the iterative algorithm (1.11) and the algorithms in [13,14] generate a sequence $\{x_n\}$ by projecting x_0 onto the intersection of closed convex sets C_n and Q_n for each $n \ge 1$ which is not easy to compute.

It is our purpose, in this paper to prove strong convergence of Ishikawa type scheme to a uniformly L-Lipschitzian and asymptotically pseudocontractive mappings in the intermediate sense without the use of hybrid method. As a consequence, Questions 1 and 2 are answered in the affirmative. No compactness assumption is imposed either on T or on C. Furthermore, projection of x_0 onto the intersection of closed convex sets C_n and Q_n for each $n \ge 1$ is not required. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear mappings.

In order to prove our results, we need the following definition and lemmas.

Let H be a real Hilbert space. The function $\phi : H \times H \to \mathbb{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \quad \text{for any } x, y \in E, \tag{1.12}$$

was studied by Alber [15], Kamimula and Takahashi [16] and Riech [17]. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2 \quad \text{for any } x, y \in E. \tag{1.13}$$

The function ϕ has also the following property:

$$\phi(y, x) = \phi(z, x) + \phi(y, z) + 2\langle z - y, x - z\rangle \quad \text{for all } x, y, z \in E. \tag{1.14}$$

Lemma 1.1. *Let H be a real Hilbert space. Then the following equality holds:*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2,$$

for all $\alpha \in (0, 1)$ and $x, y \in H$.

Lemma 1.2 *([18]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1+\gamma_n)a_n + \sigma_n, \quad n \ge n_0,$$

where, n_0 is some nonnegative integer. If $\sum \gamma_n < \infty$ and $\sum |\sigma_n| < \infty$. Then, $\lim_{n\to\infty} a_n$ exists.

2. Convergence theorem for asymptotically pseudocontractive mappings in the intermediate sense

In this section we provide convergence theorem for asymptotically pseudocontractive mappings in the intermediate sense by the Ishikawa type method.

Theorem 2.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be uniformly L-Lipschitzian and asymptotically pseudocontractive mapping in the intermediate sense with sequences $\{k_n\} \subset [1, \infty)$ and $\{\nu_n\} \subset [0, \infty)$ defined as in (1.8). Assume that the interior of $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \beta_n T^n x_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n T^n y_n + (1-\alpha_n)x_n, \quad n \ge 1, \end{cases} \tag{2.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Assume that the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$, *where* $q_n = 2k_n - 1$;
(ii) $a \le \alpha_n \le \beta_n \le b$ *for some* $a > 0$ *and some* $b \in (0, L^{-2}[\sqrt{1+L^2} - 1])$.

Then the sequence $\{x_n\}$ generated by (2.1) converges strongly to a fixed point of T.

Proof. Fix $x^* \in F$. From (2.1), Lemma 1.1 and (1.9), we have that

$$\begin{aligned}\|y_n - x^*\|^2 &= \|\beta_n(T^n x_n - x^*) + (1-\beta_n)(x_n - x^*)\|^2 \\ &= \beta_n\|T^n x_n - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2 \\ &\le \beta_n(q_n\|x_n - x^*\|^2 + \|x_n - T^n x_n\|^2 + 2\nu_n) + (1-\beta_n)\|x_n - x^*\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2 \\ &\le q_n\|x_n - x^*\|^2 + \beta_n^2\|T^n x_n - x_n\| + 2\nu_n, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\|y_n - T^n y_n\|^2 &= \|\beta_n(T^n x_n - T^n y_n) + (1-\beta_n)(x_n - T^n y_n)\|^2 \\ &= \beta_n\|T^n x_n - T^n y_n\|^2 + (1-\beta_n)\|x_n - T^n y_n\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2 \\ &\le \beta_n^3 L^2\|x_n - T^n x_n\|^2 + (1-\beta_n)\|x_n - T^n y_n\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2. \end{aligned} \tag{2.3}$$

Then from (1.9), (2.2) and (2.3), we obtain that

$$\begin{aligned}\|T^n y_n - x^*\|^2 &\le q_n\|y_n - x^*\|^2 + \|y_n - T^n y_n\|^2 + 2\nu_n, \\ &\le q_n^2\|x_n - x^*\|^2 - \beta_n(1 - q_n\beta_n - \beta_n^2 L^2 - \beta_n)\|T^n x_n - x_n\|^2 \\ &\quad + 2(q_n+1)\nu_n + (1-\beta_n)\|x_n - T^n y_n\|^2. \end{aligned} \tag{2.4}$$

Thus, from (2.1), Lemma 1.1, (2.4) and property (ii) we get that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(T^n y_n - x^*) + (1-\alpha_n)(x_n - x^*)\|^2 \\ &= \alpha_n\|T^n y_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|T^n y_n - x_n\|^2 \\ &\le \alpha_n q_n^2\|x_n - x^*\|^2 - \alpha_n\beta_n(1 - q_n\beta_n - \beta_n^2 L^2 - \beta_n) \\ &\quad \times \|T^n x_n - x_n\|^2 + 2\alpha_n(q_n+1)\nu_n + \alpha_n(1-\beta_n)\|x_n - T^n y_n\|^2 \\ &\quad + (1-\alpha_n)\|x_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|T^n y_n - x_n\|^2 \\ &\le q_n^2\|x_n - x^*\|^2 - \alpha_n\beta_n(1 - q_n\beta_n - \beta_n^2 L^2 - \beta_n)\|T^n x_n - x_n\|^2 + 2(q_n+1)\nu_n. \end{aligned} \tag{2.5}$$

Moreover, from condition (ii), there exists n_0 such that

$$1 - q_n\beta_n - \beta_n^2 L^2 - \beta_n \ge \frac{1 - 2b - L^2 b^2}{3} > 0, \quad \forall n \ge n_0, \tag{2.6}$$

thus, we get that

$$\|x_{n+1} - x^*\|^2 \le \left[1 + (q_n^2 - 1)\right]\|x_n - x^*\|^2 + 2(q_n+1)\nu_n, \quad \forall n \ge n_0. \tag{2.7}$$

Therefore, by Lemma 1.2 we obtain that $\lim_{n\to\infty}\|x_n - x^*\|$ exists.

Furthermore, from (1.14) we also have that,

$$\phi(p, x_n) = \phi(x_{n+1}, x_n) + \phi(p, x_{n+1}) + 2\langle x_{n+1} - p, x_n - x_{n+1}\rangle.$$

This implies that

$$\langle x_{n+1} - p, x_n - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) = \frac{1}{2}\Big(\phi(p, x_n) - \phi(p, x_{n+1})\Big). \tag{2.8}$$

Moreover, since the interior of F is nonempty, there exists $p^* \in F$ and $r > 0$ such that $p^* + rh \in F$, whenever $\|h\| \le 1$. Thus, from (2.7) and (2.8) we get that

$$0 \le \langle x_{n+1} - (p^* + rh), x_n - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) + M((q_n^2 - 1) + \nu_n), \tag{2.9}$$

for some $M > 0$. Then from (2.9) and (2.8) we obtain that

$$\begin{aligned} r\langle h, x_n - x_{n+1}\rangle &\le \langle x_{n+1} - p^*, x_n - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) + M((q_n^2 - 1) + \nu_n) \\ &= \frac{1}{2}(\phi(p^*, x_n) - \phi(p^*, x_{n+1})) + M((q_n^2 - 1) + \nu_n), \end{aligned}$$

and hence

$$\langle h, x_n - x_{n+1}\rangle \le \frac{1}{2r}(\phi(p^*, x_n) - \phi(p^*, x_{n+1})) + \frac{1}{r}M((q_n^2 - 1) + \nu_n).$$

Since h with $\|h\| \leq 1$ is arbitrary, we have

$$\|x_n - x_{n+1}\| \leq \frac{1}{2r}(\phi(p^*, x_n) - \phi(p^*, x_{n+1})) + \frac{1}{r}M((q_n^2 - 1) + \nu_n).$$

So, if $n > m > n_0$, then we get that

$$\begin{aligned}
\|x_m - x_n\| &= \|x_m - x_{m+1} + x_{m+1} - \cdots - x_{n-1} + x_{n-1} - x_n\| \\
&\leq \sum_{i=m}^{n-1} \|x_i - x_{i+1}\| \\
&\leq \frac{1}{2r}\sum_{i=m}^{n-1}(\phi(p^*, x_i) - \phi(p^*, x_{i+1})) + \frac{M}{r}\sum_{i=m}^{n-1}((q_i^2 - 1) + \nu_i) \\
&= \frac{1}{2r}(\phi(p^*, x_m) - \phi(p^*, x_n)) + \frac{M}{r}\sum_{i=m}^{n-1}((q_i^2 - 1) + \nu_i).
\end{aligned}$$

But we know that $\{\phi(p^*, x_n)\}$ converges, $\sum \nu_n < \infty$ and $\sum(q_n^2 - 1) < \infty$. Therefore, we obtain that $\{x_n\}$ is a Cauchy sequence. Since H is complete there exists $x^* \in H$ such that

$$x_n \to x^* \in H. \tag{2.10}$$

Moreover, since $\{x_n\}$ is subset of C which is closed and convex we have that $x^* \in C$. Furthermore, from (2.5) and (2.6) we get that

$$\frac{a^2(1 - 2b - L^2b^2)}{3}\|T^nx_n - x_n\|^2 \leq q_n^2\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2(q_n + 1)\nu_n,$$

from which it follows that

$$\lim_{n\to\infty} \|T^nx_n - x_n\| = 0. \tag{2.11}$$

Thus, since $x_n \to x^*$ we get that $T^nx_n \to x^*$ as $n \to \infty$.

Next, we show that $\|T^nx^* - x^*\| \to 0$ as $n \to \infty$. But, since T is uniformly L-Lipschitzian and $x_n \to x^*$ as $n \to \infty$ we get that

$$\|T^nx^* - T^nx_n\| \leq L\|x^* - x_n\| \to 0, \quad \text{as } n \to \infty, \tag{2.12}$$

and hence we obtain that

$$T^nx^* \to x^* \quad \text{as } n \to \infty. \tag{2.13}$$

Now, by the continuity of T we get that $x^* = \lim_{n\to\infty}(T^nx^*) = \lim_{n\to\infty} T(T^{n-1}x^*) = T(\lim_{n\to\infty}(T^{n-1}x^*)) = T(x^*)$, that is $x^* \in F(T)$. The proof is complete. □

If in Theorem 2.1, we assume that T is asymptotically pseudocontractive mapping we get the following corollary.

Corollary 2.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be uniformly L-Lipschitzian and asymptotically pseudocontractive mappings with sequences $\{k_n\} \subset [1, \infty)$. Assume that the interior of $F(T)$ is nonempty. Then the sequence $\{x_n\}$ generated by (2.1) converges strongly to a common fixed point of T.*

Proof. The proof follows from Theorem 2.1 by letting $\nu_n := 0$ for all $n \geq 1$. □

If in Corollary 2.2, we assume that T is asymptotically nonexpansive mapping then we get the following corollary.

Corollary 2.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$. Assume that the interior of $F(T)$ is nonempty. Then the sequence $\{x_n\}$ generated by (2.1) converges strongly to a common fixed point of T.*

Proof. Since every asymptotically nonexpansive mapping is uniformly L-Lipschitzian with $L := \max_{n\geq 1}\{k_n\}$ and asymptotically pseudocontractive mapping then the conclusion follows from Corollary 2.2. □

Remark 2.4. Our results extend and unify most of the results that have been proved for this important class of nonlinear mappings. In particular, Theorem 2.1 extends Theorems 2.1 (Theorem QCK) and 2.2 of Qin et al. [12] in the sense that our convergence is either strong or does not require computation of $C_n \cap Q_n$ for each $n \geq 1$. Corollary 2.3 extends Theorem SC of Schu [7] in the sense that our convergence is without the requirement that T be completely continuous or C be compact. As a consequence, Questions 1 and 2 are answered in the affirmative.

3. Convergence theorem for asymptotically strict pseudocontractive mappings in the intermediate sense

In this section we provide convergence theorem for asymptotically strict pseudocontractive mappings in the intermediate sense by the Ishikawa type method.

Let C be a subset of a real Hilbert space H and $T : C \to C$ is said be *strict pseudocontractive* if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{3.1}$$

The class of strict pseudocontractions was introduced by the Browder and Petryshyn [19] in a real Hilbert spaces. Marino and Xu [20] proved that the fixed point set of strict pseudocontractive is closed and convex, and they also obtained a weak convergence theorem for strict pseudocontractive mappings by Mann iterative process; see [20] for more details.

T is said to be *asymptotically strict pseudocontractive* if there exists a constant $k \in [0, 1)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^nx - T^ny\|^2 \le k_n\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \quad \forall x, y \in C.$$

The class of asymptotically strict pseudocontractive was introduced by Liu [21] in 1996 (see, also [22]). Kim and Xu [23] proved that the fixed point set of asymptotically strict pseudocontractions is closed and convex.

T is said to be an *asymptotically strict pseudocontractive in the intermediate sense* if

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|T^nx - T^ny\|^2 - k_n\|x - y\|^2 - k\|(I - T^n) - (I - T^n)y\|^2\right) \le 0, \tag{3.2}$$

where $k \in [0, 1)$ and $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$. Put

$$\xi_n := \max\left\{0, \sup_{x,y\in C}\left(\|T^nx - T^ny\|^2 - k_n\|x - y\|^2 - k\|(I - T^n)x - (I - T^n)y\|^2\right)\right\}.$$

It follows that $\xi_n \to 0$ as $n \to \infty$ and (3.2) is reduced to the following:

$$\|T^nx - T^ny\|^2 \le k_n\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2 + \xi_n, \quad \forall x, y \in C. \tag{3.3}$$

The class of asymptotically strict pseudocontractive mappings in the intermediate sense was introduces by Sahu et al. [24]. They obtained a weak convergence theorem of modified Mann iterative processes for these class of mappings. A strong convergence theorem was also established in a real Hilbert space by considering the so-called hybrid projection method; see [24] for more details. But we remark that the Scheme called hybrid projection method is not easy for computation. It involves computation of intersection of closed convex sets C_n and Q_n for each $n \ge 1$. Now, we provide the following strong convergence scheme for this class of mappings which does not require computation of C_{n+1} for each $n \ge 1$.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be a uniformly L-Lipschitzian and asymptotically strict pseudocontractive mapping in the intermediate sense with sequences $\{k_n\} \subset [1, \infty)$ and $\{\xi_n\} \subset [0, \infty)$ defined as in (3.3). Assume that the interior of $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \beta_nT^nx_n + (1 - \beta_n)x_n, \\ x_{n+1} = \alpha_nT^ny_n + (1 - \alpha_n)x_n, \quad n \ge 1, \end{cases} \tag{3.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Assume that the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty} \xi_n < \infty$, $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$;
(ii) $a \le \alpha_n \le \beta_n \le b$ *for some* $a > 0$ *and some* $b \in (0, L^{-2}[\sqrt{1 + L^2} - 1])$.

Then the sequence $\{x_n\}$ generated by (3.4) converges strongly to a fixed point of T.

Proof. Note that, any uniformly L-Lipschitzian and asymptotically k-strict pseudocontractive mapping T in the intermediate sense is uniformly L-Lipschitzian and asymptotically pseudocontractive mapping in the intermediate sense with $q_n := k_n$ and $\nu_n := \frac{1}{2}\xi_n$ for all $n \ge 1$ and hence the conclusion follows from Theorem 2.1. □

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be an asymptotically strict pseudocontractive mapping with sequences $\{k_n\} \subset [1, \infty)$. Assume that the interior of $F(T)$ is nonempty. Then the sequence $\{x_n\}$ generated by (3.4) converges strongly to a fixed point of T.*

Proof. Note that, any asymptotically k-strict pseudocontractive mapping T is uniformly L-Lipschitzian, since $\|T^nx - T^ny\| \le L\|x - y\|, \forall x, y \in C$, where $L = \max\{\frac{k+\sqrt{1+(k_n-1)(1-k)}}{1-k}\}$ (see, [23]) and hence the conclusion follows from Theorem 3.1 with $\xi_n = 0$ for each $n \ge 1$. □

Remark 3.3. Corollary 3.2 extends Theorems 3.1 and 4.1 of Kim and Xu [23] and of Qin et al. [12] in the sense that our convergence is either strong or does not require computation of $C_n \cap Q_n$ for each $n \ge 1$.

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